

COCHAIN SEQUENCES AND THE QUILLEN CATEGORY OF A COCLASS FAMILY

BETTINA EICK AND DAVID J. GREEN

ABSTRACT. We introduce the concept of an infinite cochain sequence and initiate a theory of homological algebra for them. We show how these sequences simplify and improve the construction of infinite coclass families (as introduced by Eick and Leedham-Green) and how they apply in proving that almost all groups in such a family have equivalent Quillen categories. We also include some examples of infinite families of p -groups from different coclass families that have equivalent Quillen categories.

1. INTRODUCTION

Coclass theory was initiated by Leedham-Green and Newman [14]. The fundamental aim of this theory is to classify and investigate finite p -groups using the coclass as primary invariant. The infinite coclass families of finite p -groups of fixed coclass as defined by Eick and Leedham-Green [10] are considered a step towards these aims. Their definition is based on a splitting theorem for a certain type of second cohomology group.

Various interesting properties of the infinite coclass families have been determined. For example, the automorphism groups and the Schur multipliers of the groups in one family can be described simultaneously for all groups in the family, see [6, 5] and [7]. It is conjectured that almost all groups in an infinite coclass family have isomorphic mod- p cohomology rings. This conjecture is still open, but it is underlined by computational evidence obtained by Eick and King [9] and by our earlier result [8] saying that the Quillen categories of almost all groups in an infinite coclass family are equivalent. The proof of the latter theorem uses a splitting theorem for cohomology groups.

In this paper we derive a generalization of the splitting theorems obtained and used in [10] and [8], and describe the splitting at the cocycle level. Based on this, we introduce the concept of an infinite cochain sequence and take the first steps towards the development of a theory of homological algebra for them.

We then show that the infinite coclass families of [10] can be defined using the infinite cochain sequence. This way of defining the families is more explicit than the definition in [10], since it is based on cocycles rather than just cohomology

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classes. This difference is significant; for example, it is useful for the investigation of the Quillen categories of the groups in a coclass family. Further, we use the infinite cochain sequences to give a new, more conceptual proof of our main theorem in [8] on the Quillen categories of the groups in an infinite coclass family.

In final part of this paper we give some examples of groups from different coclass families with equivalent Quillen categories. Let $q = p^\ell$ for a prime p , let \mathbb{Z}_p denote the p -adic integers and consider the irreducible action of C_q on $T = \mathbb{Z}_p^{(p-1)p^{\ell-1}}$ (this is unique up to equivalence). Then $G_q = T \rtimes C_q$ is an infinite pro- p -group of coclass ℓ . For $\ell = 1$ it is the unique infinite pro- p -group of coclass 1 (or of maximal class) and for $\ell > 1$ it is an interesting example for an infinite pro- p -group of coclass ℓ .

The main line groups associated with an infinite pro- p -group G of coclass r are the infinitely many lower central series quotients $G/\gamma_i(G)$ that have coclass r ; this infinite sequence is not necessarily a coclass family itself, but it consists of finitely many different coclass families. The skeleton groups associated with an infinite pro- p -group G of coclass r form a significantly larger family of groups containing the main line groups and they play an important role in coclass theory; we refer to [13, Sec. 8.4] or [11, Sec. 3] for details.

Theorem 1.1.

- (1) *For a arbitrary, fixed prime p , the Quillen categories of almost all main line groups associated with G_p are pairwise equivalent.*
- (2) *The Quillen categories of almost all skeleton groups associated with G_9 are pairwise equivalent.*

Proof. (1): See Section 8.1 for odd p ; for $p = 2$ the main line groups are the dihedral 2-groups, and the result is well known, see e.g. [8, Sec. 9].

(2): See Section 8.2. □

Remark. Theorem 1 (1) can be made more explicit: the Quillen categories of the quotients $G_p/\gamma_i(G_p)$ are equivalent for all $i \geq p + 1$. Note that $G_3/\gamma_i(G_3)$ have isomorphic mod-3 cohomology rings for $i \in \{5, 6, 7\}$, but the cohomology ring for $i = 4$ is different, see [9].

2. INFINITE COCHAIN SEQUENCES

2.1. Preliminaries. We work throughout with the normalized standard resolution, see [12, p. 8]. That is, a cochain $f \in C^n(G, M)$ is a map $f: G^n \rightarrow M$ with the additional property that $f(g_1, \dots, g_n)$ is zero if any g_i is the identity element.

We denote the coboundary operator by $\Delta: C^n(G, M) \rightarrow C^{n+1}(G, M)$. Recall that the coboundary of a 2-cochain is given by

$$\Delta f(g_1, g_2, g_3) = {}^{g_1}f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2)$$

and more generally the coboundary of an n -cochain is

$$\begin{aligned}\Delta f(g_1, \dots, g_{n+1}) &= {}^{g_1}f(g_2, \dots, g_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ &\quad + (-1)^{n+1} f(g_1, \dots, g_n).\end{aligned}$$

The n -cocycles are the elements of

$$Z^n(G, M) = \ker(C^n(G, M) \xrightarrow{\Delta} C^{n+1}(G, M)),$$

and the n -coboundaries are the elements of

$$B^n(G, M) = \operatorname{Im}(C^{n-1}(G, M) \xrightarrow{\Delta} C^n(G, M)).$$

Since $\Delta^2 = 0$ it follows that $B^n(G, M) \subseteq Z^n(G, M)$, and we set $H^n(G, M) = Z^n(G, M)/B^n(G, M)$. Elements of $H^n(G, M)$ are called cohomology classes; if f is an n -cocycle, then its cohomology class is $f + B^n(G, M) \in H^n(G, M)$.

Remark 2.1. By transfer theory, $|G| \cdot H^n(G, M) = 0$ for all $n \geq 1$: see e.g. [3, Proposition 3.6.17].

2.2. Splitting theorems. From now on for the remainder of this section we fix the following notation.

Notation 2.2. Let G be a finite p -group with $m = \log_p(|G|)$, let R be a commutative ring, let M an RG -module and let N be a submodule of M with $\operatorname{Ann}_N(p) = \{0\}$.

We need the following generalization of [8, Theorem 7], which is itself a generalization of [10, Theorem 18].

Theorem 2.3. *We use Notation 2.2 and let $n \geq 1$ and $r \geq 2m$. Then there is a splitting*

$$H^n(G, M/p^r N) \cong H^n(G, M) \oplus H^{n+1}(G, N)$$

which is natural with respect to restriction to subgroups of G .

Notation. Projection $M \twoheadrightarrow M/p^r N$ induces maps $C^n(G, M) \twoheadrightarrow C^n(G, M/p^r N)$ of cochain modules and $H^n(G, M) \rightarrow H^n(G, M/p^r N)$ of cohomology modules. We shall denote all three maps by pro_r .

Proof. Recall that $|G| = p^m$ and define $i_r: N \rightarrow M : x \mapsto p^r x$. Consider the long exact sequence in group cohomology induced by the following short exact sequence of coefficient modules

$$0 \longrightarrow N \xrightarrow{i_r} M \xrightarrow{\operatorname{pro}_r} M/p^r N \longrightarrow 0.$$

The proof of [8, Theorem 7] readily generalizes, showing that if $n \geq 0$ and $r \geq 2m$ then

$$0 \longrightarrow H^n(G, M) \xrightarrow{\operatorname{pro}_r} H^n(G, M/p^r N) \xrightarrow{\operatorname{con}_r} H^{n+1}(G, N) \longrightarrow 0$$

is a split short exact sequence, where con_r is the connecting homomorphism. \square

We now describe how Theorem 2.3 works at the cocycle level.

Proposition 2.4. *We use Notation 2.2 and let $n \geq 1$, pick $\rho \in Z^n(G, M)$ and $\eta \in Z^{n+1}(G, N)$.*

- (1) *There is a (not necessarily unique) n -cochain $\sigma \in C^n(G, N)$ such that $\Delta(\sigma) = p^m \eta$.*
- (2) *For every $r \geq m$ and for every choice of σ in (1), the induced cochain $\text{pro}_r(\rho + p^{r-m} \sigma)$ lies in $Z^n(G, M/p^r N)$.*
- (3) *For every $r \geq 2m$ and for every choice of σ in (1), the cohomology class*

$$\text{pro}_r(\rho + p^{r-m} \sigma) + B^n(G, M/p^r N) \in H^n(G, M/p^r N)$$

is the unique class corresponding via the isomorphism of Theorem 2.3 to

$$(\rho + B^n(G, M), \eta + B^{n+1}(G, N)) \in H^n(G, M) \oplus H^{n+1}(G, N).$$

Proof. (1): $p^m \eta$ is a coboundary, since $p^m H^{n+1}(G, N) = 0$ by Remark 2.1.

(2): pro_r and Δ commute, and $\Delta(\rho + p^{r-m} \sigma) = p^r \eta$ lies in the kernel of pro_r .

(3): The proof of [8, Theorem 7] says that the component maps of the isomorphism $H^n(G, M/p^r N) \rightarrow H^n(G, M) \oplus H^{n+1}(G, N)$ are the connecting homomorphism $\text{con}_r: H^n(G, M/p^r N) \rightarrow H^{n+1}(G, N)$ and the map

$$H^n(G, M/p^r N) \xrightarrow{\pi_*} H^n(G, M/p^{r-m} N) \xrightarrow{(\text{pro}_{r-m})^{-1}} H^n(G, M),$$

where $\pi: M/p^r N \rightarrow M/p^{r-m} N$ is the projection map $x + p^r N \mapsto x + p^{r-m} N$. As

$$\pi_* \text{pro}_r(\rho + p^{r-m} \sigma) = \text{pro}_{r-m}(\rho + p^{r-m} \sigma) = \text{pro}_{r-m}(\rho),$$

the image in $H^n(G, M)$ is $\rho + B^n(G, M)$.

Recall from e.g. the proof of [13, Theorem 9.1.5] that con_r is constructed as the composition

$$Z^n(G, M/p^r N) \xrightarrow{(\text{pro}_r)^{-1}} C^n(G, M) \xrightarrow{\Delta} Z^{n+1}(G, M) \xrightarrow{(i_r)_*^{-1}} Z^{n+1}(G, N),$$

with i_r as in the proof of Theorem 2.3. So $\text{pro}_r(\rho + p^{r-m} \sigma) \mapsto \rho + p^{r-m} \sigma \mapsto 0 + p^r \eta = i_r(\eta) \mapsto \eta$. Uniqueness follows. \square

2.3. The definition of cochain sequences. Using the ideas of Proposition 2.4 we now define infinite cochain sequences.

Definition. We use Notation 2.2 and let $n \geq 1$ and $r_0 \geq 0$. We call a sequence $(\alpha_r)_{r \geq r_0}$ of cochains $\alpha_r \in C^n(G, M/p^r N)$ a *cochain sequence* if there are cochains $\rho \in C^n(G, M)$ and $\sigma \in C^n(G, N)$ and an $\omega \in \{0, 1, \dots, r_0\}$ such that

$$\alpha_r = \text{pro}_r(\rho + p^{r-\omega} \sigma) \in C^n(G, M/p^r N) \quad \text{for all } r \geq r_0.$$

Note that the cochain sequences defined by $(\rho, \sigma; r_0, \omega)$ and $(\rho', \sigma'; r'_0, \omega')$ are equal if and only if $r_0 = r'_0$ and the cochains $\text{pro}_r(\rho + p^{r-\omega}\sigma)$ and $\text{pro}_r(\rho' + p^{r-\omega'}\sigma')$ are equal as elements of $C^n(G, M/p^r N)$ for all $r \geq r_0$.

Often r_0 will be clear from the context. We then also write α_\bullet for $(\alpha_r \mid r \geq r_0)$ and $M/p^\bullet N$ for $(M/p^r N \mid r \geq r_0)$. If α_\bullet is induced from $(\rho, \sigma; r_0, \omega)$, then we also write

$$\alpha_\bullet = \text{pro}_\bullet(\rho + p^{\bullet-\omega}\sigma)$$

Notation. Often r_0 and N will be fixed from the context. We then denote $M_r = M/p^r N$, we write M_\bullet for $(M_r \mid r \geq r_0)$, and we denote with $\mathbb{C}_{r_0}^n(G, M_\bullet)$ the set of all cochain sequences which start at r_0 .

2.4. Homological algebra for cochain sequences. Our next aim is to develop some elementary homological algebra for cochain sequences.

Notation 2.5. We continue to use the Notation 2.2, imposing minor additional restrictions. We assume from now on that R is a noetherian integral domain and p a prime number, which in R is neither zero nor a unit. Further, M is a finitely generated RG -module which is free as an R -module. Then $\bigcap_r p^r M = \{0\}$ by Krull's Theorem [2, 10.17], and $\text{Ann}_N(p) = \{0\}$.

Lemma 2.6. *The set $\mathbb{C}_{r_0}^n(G, M_\bullet)$ of cochain sequences is an R -module.*

Proof. Let α_\bullet be defined by $(\rho, \sigma; r_0, \omega)$, and β_\bullet by $(\rho', \sigma'; r_0, \omega')$. Then

$$\alpha_r + \beta_r = \text{pro}_r \left(\rho + \rho' + p^{r-\ell} (p^{\ell-\omega}\sigma + p^{\ell-\omega'}\sigma') \right) \quad \text{for } \ell = \max(\omega, \omega'),$$

and so $\alpha_\bullet + \beta_\bullet$ is the cochain sequence defined by $(\rho + \rho', p^{\ell-\omega}\sigma + p^{\ell-\omega'}\sigma'; r_0, \ell)$. And for $x \in R$, $x\alpha_\bullet$ is the cochain sequence defined by $(x\rho; x\sigma; r_0, \omega)$. \square

Lemma 2.7. *Let $\alpha_\bullet \in \mathbb{C}_{r_0}^n(G, M_\bullet)$ be the cochain sequence defined by $(\rho; \sigma; r_0, \omega)$.*

- (1) *Either $\alpha_\bullet = 0$, that is $\alpha_r = 0$ for all $r \geq r_0$; or $\alpha_r \neq 0$ for all sufficiently large r .*
- (2) *$\alpha_\bullet = 0$ if and only if $\rho = 0$ in $C^n(G, M)$ and σ lies in $p^\omega C^n(G, N)$.*

Proof. If $\rho = 0$ and σ is not divisible by p^ω , then $p^{r-\omega}\sigma$ is not divisible by p^r , and so α_r is non-zero for all r . If $\rho \neq 0$ then there is k such that $\text{pro}_k(\rho) \neq 0$ in $C^n(G, M/p^k N)$, and hence $\alpha_r \neq 0$ for all $r \geq k + \omega$. \square

Notation. Let $\alpha_\bullet \in \mathbb{C}_{r_0}^n(G, M_\bullet)$. We define the *level* of α_\bullet to be the smallest value of ω such that α_\bullet is defined by $(\rho, \sigma; r_0, \omega)$ for some ρ, σ .

Remark. By definition, the cochain sequence defined by $(\rho, \sigma; r_0, \omega)$ has level at most ω . Note that $(\rho, \sigma; r_0, \omega)$ and $(\rho, p\sigma; r_0, \omega + 1)$ define the same cochain sequence. Thus the level of the cochain sequence defined by $(\rho, \sigma; r_0, \omega)$ can be strictly smaller than ω .

Definition. Define $\mathbb{C}_{r_0}^n(G, M_\bullet) \xrightarrow{\Delta} \mathbb{C}_{r_0}^{n+1}(G, M_\bullet)$ by $(\Delta\alpha)_r := \Delta(\alpha_r)$. That is, if α_\bullet is defined by $(\rho, \sigma; r_0, \omega)$, then $\Delta(\alpha_\bullet)$ is defined by $(\Delta(\rho), \Delta(\sigma); r_0, \omega)$. Further, write $\mathbb{Z}_{r_0}^n(G, M_\bullet) = \ker(\Delta)$ and $\mathbb{B}_{r_0}^{n+1}(G, M_\bullet) = \text{Im}(\Delta)$.

The map Δ is R -linear and satisfies $\Delta^2 = 0$. Thus $\mathbb{B}_{r_0}^n(G, M_\bullet) \subseteq \mathbb{Z}_{r_0}^n(G, M_\bullet)$.

Remark. By Lemma 2.7 we have $\Delta(\alpha_\bullet) = 0$ if and only if $\Delta(\rho) = 0$ and $\Delta(\sigma)$ is divisible by p^ω . So we may rephrase Proposition 2.4 as follows:

Corollary 2.8. *Let $n \geq 1$ and $r_0 \geq 2m$. For every $\bar{\rho} \in H^n(G, M)$ and every $\bar{\eta} \in H^{n+1}(G, N)$ there is a cocycle sequence $\alpha_\bullet \in \mathbb{Z}_{r_0}^n(G, M_\bullet)$ of level at most m such that for every $r \geq r_0$ the cohomology class $\alpha_r + B^n(G, M_r) \in H^n(G, M_r)$ corresponds under the isomorphism of Theorem 2.3 to $(\bar{\rho}, \bar{\eta}) \in H^n(G, M) \oplus H^{n+1}(G, N)$. \square*

Lemma 2.9. *Let $n \geq 1$. Suppose that $\alpha_\bullet \in \mathbb{Z}_{r_0}^n(G, M_\bullet)$ has level $\omega \leq r_0 - m$. The following statements are equivalent:*

- (1) $\alpha_{r_1} \in B^n(G, M_{r_1})$ for some value $r_1 \geq r_0$ of r .
- (2) $\alpha_r \in B^n(G, M_r)$ for every $r \geq r_0$.
- (3) $\alpha_\bullet \in \mathbb{B}_{r_0}^n(G, M_\bullet)$.
- (4) $\alpha_\bullet = \Delta(\beta_\bullet)$ for some $\beta_\bullet \in \mathbb{C}_{r_0}^{n-1}(G, M_\bullet)$ of level at most $\omega + m$.

Proof. The implications (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) are clear. (1) \Rightarrow (4): Let $\alpha_\bullet = \text{pro}_\bullet(\rho + p^{\bullet-\omega}\sigma)$. Since $\alpha_{r_1} \in B^n(G, M_{r_1})$ there are $\phi \in C^{n-1}(G, M)$ and $\psi \in C^n(G, N)$ such that $\rho + p^{r_1-\omega}\sigma = \Delta(\phi) + p^{r_1}\psi$, and hence

$$p^{r_1-\omega}(\sigma - p^\omega\psi) = \Delta(\phi) - \rho.$$

By Lemma 2.7 we may replace σ by $\sigma - p^\omega\psi$ without altering α_\bullet . Hence

$$p^{r_1-\omega}\sigma = \Delta(\phi) - \rho.$$

Now, the right hand side is a cocycle, since $\alpha_\bullet \in \mathbb{Z}^n(G, M_\bullet)$ means that ρ is a cocycle. Hence the left hand side lies in $Z^n(G, N)$. So $\sigma \in Z^n(G, N)$ by regularity of p , and therefore since $p^m H^n(G, N) = 0$ there is $\chi \in C^{n-1}(G, N)$ with $\Delta(\chi) = p^m\sigma$. Hence $\rho = \Delta(\lambda)$ for $\lambda = \phi - p^{r_1-\omega-m}\chi \in C^{n-1}(G, M)$. So $\alpha_\bullet = \Delta(\beta_\bullet)$ for $\beta_\bullet = \text{pro}_\bullet(\lambda + p^{\bullet-\omega-m}\chi)$. \square

The next result will be needed in the proof of Lemma 4.4.

Lemma 2.10. *Let $n \geq 1$ and $r_1 \geq r_0 \geq 2m$. For each $z \in Z^n(G, M_{r_1})$ there is an $\alpha_\bullet \in \mathbb{Z}_{r_0}^n(G, M_\bullet)$ of level at most m with $\alpha_{r_1} = z$.*

Proof. Let ξ be the element of $H^n(G, M) \oplus H^{n+1}(G, N)$ corresponding to $z + B^n(G, M_{r_1}) \in H^n(G, M_{r_1})$ under the isomorphism of Theorem 2.3. By Corollary 2.8 there is some $\beta_\bullet = \text{pro}_\bullet(\rho + p^{\bullet-m}\sigma) \in \mathbb{Z}_{r_0}^n(G, M_\bullet)$ such that $\beta_r + B^n(G, M_r)$ corresponds to ξ for every $r \geq r_0$. Hence $z - \beta_{r_1} \in B^n(G, M_{r_1})$. Pick $\lambda \in C^{n-1}(G, M_{r_1})$ with $z = \Delta(\lambda) + \beta_{r_1}$, and choose $\bar{\lambda} \in C^{n-1}(G, M)$ with $\text{pro}_{r_1}(\bar{\lambda}) = \lambda$. For $\rho' = \rho + \Delta(\bar{\lambda}) \in Z^n(G, M)$ we then have $z = \alpha_{r_1}$ for $\alpha_\bullet = \text{pro}_\bullet(\rho' + p^{\bullet-m}\sigma)$. \square

3. COCLASS FAMILIES OF p -GROUPS

Coclass families are certain infinite families of finite p -groups of fixed coclass. Their construction has been introduced in [10] based on a version of Theorem 2.3. Here we exhibit a construction based on Proposition 2.4. The construction differs from [10] in that it uses cocycles rather than their corresponding cocycle classes and thus is slightly more explicit. This difference will be essential in our later applications.

Every coclass family of p -groups of coclass r is associated with an infinite pro- p -group S of coclass r . The structure of the infinite pro- p -groups of finite coclass is well investigated. For example, it is known that for each such group S there exists natural numbers l and d so that the l -th term of the lower central series $\gamma_l(S)$ satisfies that $\gamma_l(S) \cong \mathbb{Z}_p^d$, where \mathbb{Z}_p denotes the p -adic numbers, and $S/\gamma_l(S)$ is a finite p -group of coclass r . The integer d is an invariant of S called the *dimension* of S . The integer l is not an invariant; in fact, each integer larger than l can be used instead of l . The subgroup $\gamma_l(S)$ is often denoted by T and called a *translation subgroup* of S . Its subgroup series defined by $T_0 = T$ and $T_{i+1} = [T_i, S]$ satisfies $[T_i : T_{i+1}] = p$ for $i \in \mathbb{N}_0$. Thus the series $T = T_0 > T_1 > \dots$ is the unique series of S -normal subgroups in T , and T is called a *uniserial S -module*. We refer to [13] for many more details on the structure of the infinite pro- p -groups of coclass r .

Given S and l , we write $S_i = S/\gamma_{l+i}(S)$ for $i \in \mathbb{N}_0$. Then S_0, S_1, \dots is an infinite sequence of finite p -groups of coclass r . This sequence is called the *main line* associated with S . The main line is not necessarily a coclass family itself, but it always consists of d coclass families and finitely many other groups. More precisely, there exists an integer $h \geq l$ so that the d infinite sequences $(S_{h+i}, S_{h+i+d}, S_{h+i+2d}, \dots)$ for $0 \leq i < d$ are coclass families. Note that the group S can be viewed as an extension of S_{h+i+jd} by its natural module T_{h+i+jd} for each h, i and j and the group $S_{h+i+jd+k}$ can be viewed as an extension of S_{h+i+jd} by its natural module $T_{h+i+jd}/T_{h+i+jd+k}$ for each h, i, j and k .

For each coclass family (G_0, G_1, \dots) associated with the infinite pro- p -group S there exists an integer k so that each group G_j is a certain extension of S_{h+i+jd} with its natural module $T_{h+i+jd}/T_{h+i+jd+k}$. The extensions can be chosen so that the main line group $S_{h+i+jd+1}$ is not a quotient of G_j . In this case the integer k is an invariant of the coclass family called its *distance* to the main line.

To describe the groups in a coclass family explicitly, it is more convenient to use a different type of extension construction. Instead of describing a group G_j in a coclass family as extension of an associated main line group S_{h+i+jd} by its natural module of fixed size p^k , we describe each G_j as an extension of a fixed main line group S_ℓ for some suitable ℓ by a module of variable size $M_j := T_\ell/T_{\ell+jd}$. It is not difficult to observe that $T_{\ell+jd} = p^j T_\ell$ and thus the group M_j is isomorphic to a direct product of d copies of cyclic groups of order p^j .

We now use Proposition 2.4 to exhibit a complete construction for a coclass family (G_0, G_1, \dots) associated with the infinite pro- p -group S of coclass r . For this purpose let $m = \log_p(S_\ell) = r + \ell - 1$ and let $j \geq 3m + 1$. Let $\rho \in Z^2(S_\ell, T_\ell)$ so that S is an extension of S_ℓ with T_ℓ via ρ .

Definition. There exists $\eta \in Z^3(S_\ell, T_\ell)$ so that G_j is an extension of S_ℓ by M_j via τ_j where $\tau_\bullet = \text{pro}_\bullet(\rho + p^{\bullet-m}\sigma)$ and $\Delta(\sigma) = p^m\eta$.

The definition of coclass families asserts that for each coclass family there exists an η yielding this family. It may happen that different cocycles η_1, η_2 yield coclass families with pairwise isomorphic groups; for example, this is the case if $\eta_1 \equiv \eta_2 \pmod{B^3(S_\ell, T_\ell)}$. We note that every $\eta \in Z^2(S_\ell, M_j)$ yields a coclass family via the above construction.

The significance of coclass families is underlined by the fact that for $(p, r) = (2, r)$ or $(p, r) = (3, 1)$ all but finitely many p -groups of coclass r are contained in a coclass family.

4. COCHAIN SEQUENCES AND ELEMENTARY ABELIANS

We now apply the results of Section 2.4 to the coclass family G_\bullet of Section 3. In the language of Notation 2.5 this means that $M = T$. It would be natural to take $N = T$ as well, but for technical reasons¹ we shall actually take $N = pT$. Hence $M_\bullet = M/p^\bullet N = T/p^{\bullet+1}T$, and $G_{\bullet+1} = M_\bullet.P$ with extension cocycle $\tau_\bullet \in \mathbb{Z}_{r_0}^2(P, M_\bullet)$.

Recall that if $N \trianglelefteq G$ and $U \leq G/N$, then a *lift* of U is a subgroup $\bar{U} \leq G$ such that the projection map $G \rightarrow G/N$ maps \bar{U} isomorphically to U .

Suppose that we are given $H \leq G_{r+1}$. Setting $K := H \cap M_r$ and $Q := HM_r/M_r$, we see that H is an extension $H = K.Q$, with $Q \leq P$, and K a Q -submodule of M_r . If K has a complement C in H – which is certainly the case if H is elementary abelian –, then $C \leq G_{r+1}$ is a lift of Q .

4.1. Extension theory. We recall some details of extension theory, see e.g. [3, §3.7]. Let G be a finite group and M a left $\mathbb{Z}G$ -module. Recall that every group extension $\Gamma = M.G$ can be constructed using a 2-cocycle $\tau \in Z^2(G, M)$: the underlying set is $M \times G$, with multiplication

$$(t_1, g_1)(t_2, g_2) = (t_1 + {}^{g_1}t_2 + \tau(g_1, g_2), g_1g_2).$$

Associativity is equivalent to the cocycle condition. The extension splits as a semidirect product $\Gamma = M \rtimes G$ if and only if $\tau \in B^2(G, M)$. If $\tau = \Delta(f)$ then $G(f) = \{(-f(g), g) \mid g \in G\} \leq \Gamma$ is a lift of G , and every lift arises thus.

¹It simplifies Remark 4.3 and especially Lemma 5.1 if M/p^rN and M/N have the same elementary abelian subgroups.

Lemma 4.1. *If $f, f' \in C^1(G, M)$ satisfy $\Delta(f) = \tau = \Delta(f')$ then $f' - f \in Z^1(G, M)$ and moreover*

$$f' - f \in B^1(G, M) \iff G(f) \text{ and } G(f') \text{ are conjugate by an element of } M.$$

Proof. This is Proposition 3.7.2 of [3]. Observe from the proof of that result that conjugation by elements of M induces every coboundary. \square

4.2. Lifting elementary abelians.

Lemma 4.2. *Let $Q \leq P$ and suppose $r_0 \geq 2m$. Then the three following statements are equivalent:*

- (1) Q has a lift $\bar{Q}_r \leq G_{r+1}$ for all $r \geq r_0$.
- (2) Q has a lift $\bar{Q}_r \leq G_{r+1}$ for at least one $r \geq r_0$.
- (3) $\tau_\bullet|_Q = \Delta(f_\bullet)$ for some cochain sequence $f_\bullet \in \mathbb{C}_{r_0}^1(Q, M_\bullet)$ of level at most m .

Proof. Let $H_r \leq G_{r+1}$ be the subgroup with $M_r \leq H_r$ and $H_r/M_r = Q$. Then $H_r = M_r.Q$ with extension class $\tau_r|_Q$, and Q has a lift $\bar{Q}_r \leq G_{r+1}$ if and only if $\tau_r|_Q$ lies in $B^2(Q, M_r)$. Now apply Lemma 2.9. \square

Now suppose that $E \leq G_{r+1}$ is elementary abelian. Setting $K = E \cap M_r$ and $U = EM_r/M_r \leq P$ as above, we have $E = K \times \bar{U}$ for a lift $\bar{U} \leq G_{r+1}$ of U . Recall that $\bar{U} = U(\phi)$ for some $\phi \in C^1(U, M_r)$ with $\Delta(\phi) = \tau_r|_U$.

Notation. We shall need to refer to several different maps between cohomology modules. Let $L \subseteq M$ be a submodule.

- Inclusion $L \hookrightarrow M$ induces $H^n(G, L) \xrightarrow{\text{inc}} H^n(G, M)$.
- $\text{mul}^r : M/L \rightarrow M/p^r L$, $x + L \mapsto p^r x + p^r L$ induces $H^n(G, M/L) \xrightarrow{\text{mul}^r} H^n(G, M/p^r L)$.

Note that $\text{mul}^r \circ \text{mul}^s = \text{mul}^{r+s}$, and $\text{pro}_r(\rho + p^{r-m}\sigma) = \text{pro}_r(\rho) + \text{mul}^{r-m} \text{pro}_m(\sigma)$.

Remark 4.3. Since $K \leq M_r = T/p^{r+1}T$ is elementary abelian, it follows that

$$K \leq \Omega_1(M_r) = p^r T / p^{r+1} T \xrightarrow[\cong]{(\text{mul}^r)^{-1}} T/pT.$$

So $K = \text{mul}^r(W)$ for some $W \leq T/pT$. Since E is abelian, we have $[\bar{U}, K] = 1$, which is equivalent to $W \leq (T/pT)^U$.

Notation. \mathcal{E} is the set of all triples (U, f_\bullet, W) with $U \leq P$ an elementary abelian; $f_\bullet \in \mathbb{C}_{r_0}^1(U, M_\bullet)$ a cochain sequence of level at most $2m$ such that $\Delta(f_\bullet) = \tau_\bullet|_U$; and $W \leq (T/pT)^U$.

Lemma 4.4. *Suppose that $r \geq r_0 \geq 2m$. Every elementary abelian $E \leq G_{r+1}$ has the form $E = E_r(U, f_\bullet, W) := \text{mul}^r(W) \times U(f_r)$ for some $(U, f_\bullet, W) \in \mathcal{E}$.*

Proof. We saw above that $E = \text{mul}^r(W) \times U(\phi)$ with $U \leq P$ elementary abelian; $W \leq (T/pT)^U$; and $\phi \in C^1(U, M_r)$ with $\Delta(\phi) = \tau_r|_U$. As $U(\phi)$ is a lift of U in G_{r+1} , there is $f_\bullet \in \mathbb{C}_{r_0}^1(U, M_\bullet)$ of level at most $2m$ with $\Delta(f_\bullet) = \tau_\bullet|_U$ by Lemma 4.2. Hence $\phi - f_r \in Z^1(U, M_r)$, so by Lemma 2.10 there is $z_\bullet \in \mathbb{Z}_{r_0}^1(U, M_\bullet)$ of level at most m with $z_r = \phi - f_r$. So $(U, f_\bullet + z_\bullet, W) \in \mathcal{E}$, and $E = \text{mul}^r(W) \times U(f_r + z_r)$. \square

5. CHANGE OF MODULE

The following technical lemma is required in the proofs of Proposition 6.1 and Lemma 7.1. We revert to Notation 2.5, and consider the case of two submodules $L, N \subseteq M$ satisfying the condition $pL \subseteq N \subseteq L$.

Example. If $(U, f_\bullet, W) \in \mathcal{E}$, then $W \leq T/pT$, and so $W = L/pT$ for some $pT \subseteq L \subseteq T$. Hence $pL \subseteq N \subseteq L$, since $N = pT$.

We shall investigate the cohomology maps induced by the short exact sequence

$$0 \rightarrow L/N \xrightarrow{\text{mul}^r} M/p^r N \rightarrow M/p^r L \rightarrow 0.$$

As we now have to distinguish between two different projection maps, we shall denote them by $M \xrightarrow{\text{pro}_r^N} M/p^r N$ and $M \xrightarrow{\text{pro}_r^L} M/p^r L$.

Lemma 5.1. *Suppose the RG -submodule $L \subseteq M$ satisfies $pL \subseteq N \subseteq L$.*

- (1) *Assume $r_0 \geq 1$. Then $j_* \circ i_* = 0$ for the chain maps*

$$C^*(G, L/N) \xrightarrow{i_*} \mathbb{C}_{r_0}^*(G, M/p^\bullet N) \xrightarrow{j_*} \mathbb{C}_{r_0}^*(G, M/p^\bullet L)$$

given by $i_n(c)_r = \text{mul}^r(c)$ and $j_n(\alpha_\bullet)_r = \alpha_r + p^r L$.

- (2) *Suppose that $\alpha_\bullet \in \mathbb{Z}_{r_0}^n(G, M/p^\bullet N)$ satisfies $j_n(\alpha_\bullet) \in \mathbb{B}_{r_0}^n(G, M/p^\bullet L)$. If α_\bullet has level $\omega \leq r_0 - m$, then*

$$\alpha_\bullet = i_n(c)_\bullet + \Delta(\text{pro}_\bullet^N(\kappa + p^{\bullet-(\omega+m)}\lambda))$$

for some $c \in Z^n(G, L/N)$, $\kappa \in C^{n-1}(G, M)$ and $\lambda \in C^{n-1}(G, L)$.

Proof. (1): Pick $\bar{c} \in C^n(G, L)$ with $\text{pro}_0^L(\bar{c}) = c$, then $p\bar{c} \in C^n(G, N)$ and

$$i_n(c)_\bullet = \text{pro}_\bullet^N(0 + p^{\bullet-1} \cdot p\bar{c}) \in \mathbb{C}_{r_0}^n(G, M/p^\bullet N), \quad \text{with level } 1 \leq r_0.$$

If $\alpha_\bullet = \text{pro}_\bullet^N(\rho + p^{\bullet-\omega}\sigma) \in \mathbb{C}_{r_0}^n(G, M/p^\bullet N)$, then $j_n(\alpha_\bullet) = \text{pro}_\bullet^L(\rho + p^{\bullet-\omega}\sigma)$.

Clearly i_*, j_* are chain maps. And $j_n i_n = 0$, since c takes values in L .

(2): Let $\alpha_\bullet = \text{pro}_\bullet^N(\rho + p^{\bullet-\omega}\sigma)$, so $j_n(\alpha_\bullet) = \text{pro}_\bullet^L(\rho + p^{\bullet-\omega}\sigma)$. By Lemma 2.9 we have $j_n(\alpha_\bullet) = \Delta(\gamma_\bullet)$ for some $\gamma_\bullet \in \mathbb{C}_{r_0}^{n-1}(G, M/p^\bullet L)$ of the form

$$\gamma_\bullet = \text{pro}_\bullet^L(\kappa + p^{\bullet-\omega-m}\lambda) \quad \text{with } \kappa \in C^{n-1}(G, M), \lambda \in C^{n-1}(G, L).$$

Applying Lemma 2.7 we have $\rho = \Delta(\kappa)$, and

$$p^m \sigma = \Delta(\lambda) + p^{\omega+m} \bar{c} \quad \text{for some } \bar{c} \in C^n(G, L).$$

From $\alpha_\bullet \in \mathbb{Z}_{r_0}^n(G, M/p^\bullet N)$ it follows that $\Delta(\sigma)$ takes values in $p^\omega N$, and so $\Delta(\bar{c})$ takes values in N . So $c := \text{pro}_0^N(\bar{c})$ lies in $Z^n(G, L/N)$, and $i_n(c)_r = \text{pro}_r^N(p^r \bar{c})$. Hence

$$\begin{aligned} \alpha_\bullet &= \text{pro}_\bullet^N(\rho + p^{\bullet-\omega} \sigma) \\ &= \text{pro}_\bullet^N(\Delta(\kappa + p^{\bullet-(\omega+m)} \lambda) + p^\bullet \bar{c}) \\ &= i_n(c) + \Delta(\text{pro}_\bullet(\kappa + p^{\bullet-(\omega+m)} \lambda)). \end{aligned} \quad \square$$

6. MORPHISMS IN THE QUILLEN CATEGORY

Notation. Consider the triple $(U, f_\bullet, W) \in \mathcal{E}$. Recall from Section 2 that G_{r+1} has underlying set $M_r \times P$, and that $U(f_r) = \{(-f_r(u), u) \mid u \in U\}$. So the subgroup $E_r(U, f_\bullet, W) \leq G_{r+1}$ of Lemma 4.4 is

$$E_r(U, f_\bullet, W) = \{(p^r w - f_r(u), u) \mid u \in U, w \in W\}.$$

Let $j_r^f: W \times U \rightarrow E_r(U, f_\bullet, W)$ be the isomorphism $(w, u) \mapsto (p^r w - f_r(u), u)$.

Proposition 6.1. *Suppose $r_0 \geq 3m$ and $m \geq 1$. For $(U, f_\bullet, W), (U', f'_\bullet, W') \in \mathcal{E}$ the set of isomorphisms $W \times U \rightarrow W' \times U'$ of the form*

$$W \times U \xrightarrow{j_r^f} E_r(U, f_\bullet, W) \xrightarrow{\text{conjugation in } G_{r+1}} E_r(U', f'_\bullet, W') \xrightarrow{(j_r^{f'})^{-1}} W' \times U'$$

is independent of r .

For the proof we need two lemmas. Observe that mul^{r-r_0} embeds $M_{r_0} = T/p^{r_0+1}T$ in G_{r+1} as $p^{r-r_0}T/p^{r+1}T \leq M_r$.

Lemma 6.2. *Suppose that $m \geq 1$ and $r_0 \geq 2m$. Let $(U, f_\bullet, W) \in \mathcal{E}$.*

- (1) $\text{Aut}_{M_r}(E_r(U, f_\bullet, W)) = \text{Aut}_{\text{mul}^{r-r_0}(M_{r_0})}(E_r(U, f_\bullet, W)).$
- (2) *The subgroup*

$$\mathcal{N} = \{x \in M_{r_0} \mid \text{mul}^{r-r_0}(x) \in N_{G_{r+1}}(E_r(U, f_\bullet, W))\}$$

depends on neither r nor f_\bullet . Nor does the action of \mathcal{N} on $W \times U$ obtained by using j_r^f to identify $E_r(U, f_\bullet, W)$ with $W \times U$.

Write \bar{W} for the module $pT \subseteq \bar{W} \subseteq T$ with $W = \bar{W}/pT$.

Proof. (1): Conjugation by $t \in T$ fixes $M_r U(f_r)/M_r$ and W pointwise, and is described by $\Delta(t) \in B^1(U, M_r)$:

$${}^{(t,0)}(p^r w - f_r(u), u) = (p^r w - \Delta(t)(u) - f_r(u), u).$$

If t normalizes $\text{mul}^r(W) \times U(f_r) \leq G_{r+1}$ then $\Delta(t)$ must take values in $p^r \bar{W}$. Hence $\Delta(t) \in p^r Z^1(U, \bar{W}) \subseteq p^{r-m} B^1(U, \bar{W})$. So there is $\bar{v} \in \bar{W}$ such that $\Delta(t) = p^{r-m} \Delta(\bar{v})$, and $\text{pro}_r(p^{r-m} \bar{v}) = \text{mul}^{r-r_0} \text{pro}_{r_0}(p^{r_0-m} \bar{v}) \in \text{mul}^{r-r_0}(M_{r_0})$ has the same conjugation action as t .

(2): Conversely if $v = \bar{v} + p^{r_0+1}T \in M_{r_0}$ then $\text{mul}^{r-r_0}(v)$ normalizes $\text{mul}^r(W) \times U(f_r)$ if and only if $\Delta(p^{r-r_0} \bar{v}) \in Z^1(U, T)$ takes values in $p^r \bar{W}$, that is if $\Delta(v) =$

$\text{mul}^{r_0}(z)$ for some $z \in Z^1(U, W)$. And the action on $W \times U$ is then $(w, u) \mapsto (w - z(u), u)$. \square

Lemma 6.3. *Suppose $r_0 \geq 2m$ and $g \in P$. Let $(U, f_\bullet, W), ({}^gU, f'_\bullet, {}^gW) \in \mathcal{E}$. Define $\chi_r \in C^1({}^gU, M_r)$ by*

$$\chi_r(v) = ({}^gf_r)(v) - \tau_r(g, v^g) + \tau_r(v, g) - f'_r(v).$$

Then $\chi_\bullet \in \mathbb{Z}_{r_0}^1({}^gU, M_\bullet)$ is a cocycle sequence of level at most $2m$.

Proof. For $c \in C^n(H, M)$ we of course define ${}^gc \in C^n({}^gH, M)$ by

$$({}^gc)(k_1, \dots, k_n) = {}^gc(k_1^g, \dots, k_n^g).$$

Everything is of level at most $2m$. Since $\Delta(f'_\bullet) = \tau_\bullet|_{{}^gU}$ and $\Delta({}^gf_\bullet) = {}^g(\tau_\bullet|_U)$ we have

$$\begin{aligned} \Delta(\chi_r)(v_1, v_2) &= {}^g\tau_r(v_1^g, v_2^g) - \tau_r(g, v_2^g) + \tau_r(g, v_1^g v_2^g) - \tau_r(g, v_1^g) \\ &\quad + \tau_r(v_2, g) - \tau_r(v_1 v_2, g) + \tau_r(v_1, g) - \tau_r(v_1, v_2) \\ &= \Delta(\tau_r)(g, v_1^g, v_2^g) - \Delta(\tau_r)(v_1, g, v_2^g) + \Delta(\tau_r)(v_1, v_2, g) = 0. \quad \square \end{aligned}$$

Proof of Proposition 6.1. If $({}^{t,g})E_r(U, f_\bullet, W) = E_r(U', f'_\bullet, W')$ then ${}^gU = U'$ and ${}^gW = W'$. So we assume that $g \in P$ is fixed, and consider which $t \in M_r$ satisfy $({}^{t,g})E_r(U, f_\bullet, W) = E_r({}^gU, f'_\bullet, {}^gW)$, and which isomorphisms arise in this way.

The map $F: W \times U \rightarrow {}^gW \times {}^gU$ given by j_r^f , then conjugation by (t, g) , and then $(j_r^{f'})^{-1}$ must have the form $F(w, u) = ({}^gw - \pi({}^gu), {}^gu)$ for some $\pi \in Z^1({}^gU, {}^gW)$. So we consider g, π to be fixed and ask for which values of r there is $t + p^{r+1}T \in M_r$ realising this F . The condition on t, i can be phrased thus:

$$(t, g)(p^r w - f_r(u), u) = (p^r \cdot {}^gw - p^r \pi({}^gu) - f'_r({}^gu), {}^gu)(t, g)$$

Equality in gU is immediate. We are left with the following condition in M_r :

$$t + p^r \cdot {}^gw - {}^g(f_r(u)) + \tau_r(g, u) = p^r ({}^gw - \pi({}^gu)) - f'_r({}^gu) + ({}^gu)t + \tau_r({}^gu, g).$$

So with χ_r as in Lemma 6.3 we have

$$\begin{aligned} p^r \pi({}^gu) &= ({}^gf)({}^gu) - \tau_r(g, u) + \tau_r({}^gu, g) - f'_r({}^gu) + \Delta(t)({}^gu) \\ &= (\chi_r + \Delta(t))({}^gu). \end{aligned}$$

That is, F is realisable for this r if and only if

$$p^r \pi - \chi_r \in B^1({}^gU, M_r). \quad (1)$$

Since π takes values in ${}^gW = {}^g\bar{W}/pT$, a necessary condition for any such F to be realisable is that

$$\chi_r + p^r \cdot {}^g\bar{W} \in B^1({}^gU, T/p^r \cdot {}^g\bar{W}). \quad (2)$$

Since $\chi_\bullet \in \mathbb{Z}_{r_0}^1({}^gU, M_\bullet)$ has level at most $2m$ and $r_0 \geq 2m + m$, we deduce from Lemma 2.9 that (2) is either satisfied for all r_0 , or for none.

If (2) is satisfied then we apply Lemma 5.1 with $G = {}^gU$, $\alpha_\bullet = \chi_\bullet$ and $L = {}^g\bar{W}$, hence $L/N = {}^gW$. Note that χ_\bullet has level at most $2m \leq r_0 - m$. We conclude

that there are $c \in Z^1({}^gU, {}^gW)$, $\kappa \in C^0({}^gU, T)$ and $\lambda \in C^0({}^gU, {}^g\bar{W})$ with $\chi_\bullet = \text{mul}^\bullet(c) + \Delta(\text{pro}_\bullet(\kappa + p^{\bullet-3m}\lambda))$. We conclude that if we take $\pi = c$ then Eqn. (1) is solvable for all r . That is, this one map $F: W \times U \rightarrow {}^gW \times {}^gU$ is independent of r . But all other maps for this value of g correspond to a M_r -automorphism of $U \times W$ followed by F , and we saw in Lemma 6.2 that these isomorphisms are independent of r too. \square

Corollary 6.4. *Suppose $e \geq 3m$ and $m \geq 1$. For $(U, f_\bullet, W), (U', f'_\bullet, W') \in \mathcal{E}$ the following statements are equivalent:*

- (1) $E_r(U, f_\bullet, W)$ and $E_r(U', f'_\bullet, W')$ are G_{r+1} -conjugate for some r .
- (2) $E_r(U, f_\bullet, W)$ and $E_r(U', f'_\bullet, W')$ are G_{r+1} -conjugate for every r .

Proof. They are G_{r+1} -conjugate if and only if the set of isomorphisms in Proposition 6.1 is non-empty. But this set does not depend on r . \square

7. WRAPPING UP THE MAIN THEOREM

Lemma 7.1. *Suppose $r_0 \geq 2m$. Let $(U, f_\bullet, W) \in \mathcal{E}$. For each $V \leq W \times U$ there is $(U', f'_\bullet, W') \in \mathcal{E}$ such that $\forall r : j_r^f(V) = E_r(U', f'_\bullet, W')$. Moreover the map*

$$\kappa^V : W' \times U' \xrightarrow{j_r^{f'}} E_r(U', f'_\bullet, W') \hookrightarrow E_r(U, f_\bullet, W) \xrightarrow{(j_r^f)^{-1}} W \times U$$

has image V and is independent of r .

Proof. Take $W' = V \cap W$ and $U' = \{u \in U \mid uW \in VW/W\} \leq U$. Then $V/W' \cong U'$ and W' is a direct factor of V , so there is $c \in Z^1(U', W)$ with $V = \{(w - c(u), u) \mid w \in W', u \in U'\}$. Then

$$j_r^f(V) = \{(p^{i+e-1}w - p^{i+e-1}c(u) - f_r(u), u) \mid w \in W', u \in U'\}.$$

Done with $f'_\bullet = f_\bullet|_{U'} + i_1(c)_\bullet$ in the terminology of Lemma 5.1. In particular, $\kappa^V(w, u) = (w - c(u), u)$. \square

Corollary 7.2. *Suppose $r_0 \geq 3m$ and $m \geq 1$. For $(U, f_\bullet, W), (U', f'_\bullet, W') \in \mathcal{E}$ the set of homomorphisms $W \times U \rightarrow W' \times U'$ of the form*

$$W \times U \xrightarrow{j_r^f} E_r(U, f_\bullet, W) \xrightarrow{\text{morphism in } \mathcal{A}_p(G_{r+1})} E_r(U', f'_\bullet, W') \xrightarrow{(j_r^{f'})^{-1}} W' \times U'$$

is independent of r .

Proof. Every such map is an isomorphism to some $V \leq W' \times U'$. Lemma 7.1: For some (U'', f''_\bullet, W'') have $j_r^{f'}(V) = E_r(U'', f''_\bullet, W'')$ for all r . Proposition 6.1: The set \mathcal{I}_V of isomorphisms of the form

$$W \times U \xrightarrow{j_r^f} E_r(U, f_\bullet, W) \xrightarrow{\text{conjugation in } G_{r+1}} E_r(U'', f''_\bullet, W'') \xrightarrow{(j_r^{f'})^{-1}} W'' \times U''$$

is independent of r . But $\phi \mapsto \kappa^V \circ \phi$ is a bijection from \mathcal{I}_V to the set of homomorphisms of the form

$$W \times U \xrightarrow{j_r^f} E_r(U, f_\bullet, W) \xrightarrow{\text{morphism in } \mathcal{A}_p(G_{r+1})} E_r(U', f'_\bullet, W') \xrightarrow{(j_r^{f'})^{-1}} W' \times U'$$

whose image is V . □

Proposition 7.3. *Suppose $e \geq 3m$ and $m \geq 1$. Choose a subset $\mathcal{E}_0 \subseteq \mathcal{E}$ such that for every conjugacy class C of elementary abelian subgroups in G_{r_0+1} there is exactly one $(U, f_\bullet, W) \in \mathcal{E}_0$ such that $E_{r_0}(U, f_\bullet, W)$ lies in C . Define \mathcal{C}_r to be the full subcategory of the Quillen category $\mathcal{A}_p(G_{r+1})$ on the $E_r(U, f_\bullet, W)$ with (U, f_\bullet, W) in \mathcal{E}_0 . Then:*

- (1) \mathcal{C}_r is a skeleton of $\mathcal{A}_p(G_{r+1})$ for every $r \geq r_0$.
- (2) The categories \mathcal{C}_r are all isomorphic to each other.

Hence the Quillen categories $\mathcal{A}_p(G_{r+1})$ are all equivalent to each other.

Proof. \mathcal{E}_0 exists by Lemma 4.4. (1): Need to show that each conjugacy class C in G_{r+1} contains $E_r(U, f_\bullet, W)$ for precisely one $(U, f_\bullet, W) \in \mathcal{E}_0$. Corollary 6.4: at most one such triple. Lemma 4.4: there is some $(U', f'_\bullet, W') \in \mathcal{E}$ such that $E_r(U', f'_\bullet, W')$ lies in C . By construction of \mathcal{E}_0 there is $(U, f_\bullet, W) \in \mathcal{E}_0$ such that $E_0(U, f_\bullet, W), E_0(U', f'_\bullet, W')$ are G_0 -conjugate. So $E_r(U, f_\bullet, W)$ lies in C by Corollary 6.4.

(2): For $r, r' \geq r_0$ and $(U, f_\bullet, W) \in \mathcal{E}_0$ have isomorphism

$$\lambda_{rr'}^f: E_r(U, f_\bullet, W) \xrightarrow{(j_r^f)^{-1}} W \times U \xrightarrow{j_{r'}^f} E_{r'}(U, f_\bullet, W),$$

mit $\lambda_{r'r}^f = (\lambda_{rr'}^f)^{-1}$. For a morphism $E_r(U, f_\bullet, W) \xrightarrow{\phi} E_r(U', f'_\bullet, W')$ in \mathcal{C}_r , define $F(\phi)$ in $\mathcal{C}_{r'}$ thus:

$$F(\phi): E_{r'}(U, f_\bullet, W) \xrightarrow{\lambda_{rr'}^f} E_r(U, f_\bullet, W) \xrightarrow{\phi} E_r(U', f'_\bullet, W') \xrightarrow{\lambda_{r'r'}^{f'}} E_{r'}(U', f'_\bullet, W').$$

This is a bijection

$$\mathcal{C}_r(E_r(U, f_\bullet, W), E_r(U', f'_\bullet, W')) \rightarrow \mathcal{C}_{r'}(E_{r'}(U, f_\bullet, W), E_{r'}(U', f'_\bullet, W'))$$

by Corollary 7.2, and it is functorial since $F(\text{Id}_{W \times U_r(f)}) = \lambda_{rr'}^f \text{Id} \lambda_{r'r}^f = \text{Id}_{W \times U_{r'}(f)}$, and for $E_r(U', f'_\bullet, W') \xrightarrow{\psi} E_r(U'', f''_\bullet, W'')$ in \mathcal{C}_r have

$$F(\psi)F(\phi) = \lambda_{rr'}^{f''} \psi \lambda_{r'r}^{f'} \circ \lambda_{ir'}^{f'} \phi \lambda_{r'r}^f = \lambda_{rr'}^{f''} \psi \phi \lambda_{r'r}^f = F(\psi\phi).$$

This establishes (2). The last part follows from (1) and (2). □

8. EXAMPLES

8.1. Main line maximal class groups. If p is an odd prime then $\mathcal{G}(p, 1)$ consists of one infinite tree, together with the isolated point C_{p^2} : so there is only one uniserial p -adic space group of coclass one. We recall the construction of the main line groups from Example 3.1.5(ii) of [13].

The p th local cyclotomic number field is $K = \mathbb{Q}_p(\theta)$, where θ has minimal polynomial $\Phi_p(X) = \frac{X^p-1}{X-1}$. The ring of integers in K is $\mathcal{O} = \mathbb{Z}_p[\theta]$, a free \mathbb{Z}_p -module of rank $p-1$ with basis $1, \theta, \dots, \theta^{p-2}$. The coclass one uniserial p -adic

space group is then $G := \mathcal{O} \rtimes C_p$, where the generator τ of C_p acts as multiplication by θ ; that is, $\tau v = \theta v$ for $v \in \mathcal{O}$.

The valuation ring \mathcal{O} has unique maximal ideal $\alpha\mathcal{O}$, where $\alpha = \theta - 1$. So $\gamma_i(G) = \alpha^{i-1}\mathcal{O}$ for $i \geq 2$; and by considering $\Phi_p(X + 1)$ one observes that $p\mathcal{O} = \alpha^{p-1}\mathcal{O}$. Since $1, \alpha, \dots, \alpha^{p-2}$ is a \mathbb{Z}_p -basis of \mathcal{O} , this means that $\mathcal{O}/\alpha\mathcal{O} \cong \mathbb{F}_p$, and hence $\gamma_i(G)/\gamma_{i+1}(G) \cong C_p$ for $i \geq 2$.

The main line groups are the quotients $G_i = G/\gamma_i(G)$. These main line groups fall into $p - 1$ coclass families, where for $0 \leq r \leq p - 2$ the i th group in the r th family is $G_{r+(p-1)i}$. From [8] (see also Proposition 7.3) we know that all groups in one coclass family have equivalent Quillen categories. But here a stronger result holds: all $p - 1$ coclass families have the same equivalence class of Quillen categories.

Lemma 8.1. *For this group $G = \mathcal{O} \rtimes C_p$, the Quillen category of $G/\gamma_i(G)$ is independent (up to equivalence of categories) of i for $i \geq p + 1$.*

Remark. For $p = 3$, the first author and S. King [9] have shown that $G/\gamma_5(G)$, $G/\gamma_6(G)$ and $G/\gamma_7(G)$ have isomorphic cohomology rings; and that these differ from the cohomology ring of $G/\gamma_4(G) \cong 3_+^{1+2}$.

Proof. If $v \in \mathcal{O}$ then $(v\tau)^p = (\Phi_{p-1}(\theta) \cdot v)\tau^p = 1$, and so $v\tau$ has order p . So since $p\mathcal{O} = \alpha^{p-1}\mathcal{O}$, there are two kinds of order p elements of $G/\gamma_i(G)$:

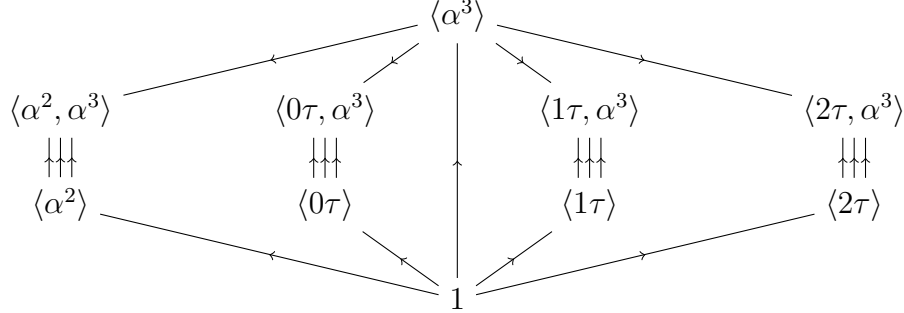
- Elements of the form $v\tau^r\gamma_i(G)$, with $v \in \mathcal{O}$ and $1 \leq r \leq p - 1$; and
- Elements of $\gamma_{i-p+1}(G)/\gamma_i(G) \cong (C_p)^{p-1}$.

Moreover the conjugacy class of $v\tau$ in G is $\{w\tau \mid w \in v + \alpha\mathcal{O}\}$; and the centralizer of $v\tau\gamma_i(G)$ in $G/\gamma_i(G)$ is elementary abelian of rank 2, generated by $v\tau\gamma_i(G)$ and $\gamma_{i-1}(G)/\gamma_i(G)$. So as τ acts on $\gamma_{i-p+1}(G)/\gamma_i(G) = \alpha^{i-p}\mathcal{O}/\alpha^{i-1}\mathcal{O}$ as multiplication by $1 + \alpha$, the objects of the Quillen category form the following equivalence classes:

- The class of $\langle v_j\tau\gamma_i(G), \gamma_{i-1}(G)/\gamma_i(G) \rangle \cong C_p^2$ for some fixed transversal v_1, \dots, v_p of $\mathcal{O}/\alpha\mathcal{O}$;
- The class of $\langle v_j\tau\gamma_i(G) \rangle \cong C_p$ for the same transversal v_1, \dots, v_p ; and
- The conjugacy classes of subgroups of $\gamma_{i-p+1}(G)/\gamma_i(G) \cong \mathcal{O}/\alpha^{p-1}\mathcal{O} \cong C_p^{p-1}$ under the action of C_p given by multiplication by $1 + \alpha$.

So the equivalence classes of objects admit a description which is independent of i . From this description and the fact that $\langle v_j\tau\gamma_i(G), \gamma_{i-1}(G)/\gamma_i(G) \rangle$ has normalizer $\langle v_j\tau\gamma_i(G), \gamma_{i-2}(G)/\gamma_i(G) \rangle$, it follows that the morphisms between these representatives also admit a description which is independent of i . \square

Since $p \in \alpha\mathcal{O}$ we may always take the transversal v_1, \dots, v_p to be $0, 1, \dots, p-1$. For $p = 3$ and $i = 4$ the Quillen category has skeleton



where the three automorphisms of each rank two elementary abelian are omitted for clarity. Specifically, the three maps $\langle 2\tau \rangle \rightarrow \langle 2\tau, \alpha^3 \rangle$ are $2\tau \mapsto (2 + \lambda\alpha^3)\tau$ for $\lambda = 0, 1, 2$; and three automorphisms of $\langle 2\tau, \alpha^3 \rangle$ fix α^3 and act on 2τ as one of these three maps.

8.2. A more substantial example. Together with Leedham–Green, Newman and O’Brien, the first author studied the 3-groups of coclass two in [11]. In particular they construct the skeleton groups in the four coclass trees (out of sixteen) whose branches have unbounded depth. Here we consider the skeleton groups in one of these unbounded depth trees: the tree associated to the pro-3-group which they denote as R (see their Theorem 4.2(a)).

We briefly recall the construction of the skeleton groups $R_{j-3,\gamma,m}$ from [11, Sect. 5]. Let $j \geq 7$. Let $K = \mathbb{Q}_3(\theta)$ be the ninth local cyclotomic number field, so θ is a root of $\Phi_9(X) = X^6 + X^3 + 1$. Let \mathcal{O} be the ring of integers in K ; then $\mathcal{O} = \mathbb{Z}_3[\theta]$ is free as a \mathbb{Z}_3 -module, with basis $1, \theta, \dots, \theta^5$. Moreover, \mathcal{O} is a local ring, with maximal ideal $\mathfrak{p} = (\theta - 1)\mathcal{O}$. Observing that $(\theta - 1)^6$ and 3 are associates, one sees that $3\mathcal{O} = \mathfrak{p}^6$.

We now recall the twisting map $\mathfrak{p} \wedge \mathfrak{p} \rightarrow \mathcal{O}$, which we shall denote by γ_0 . Note however that in [11] it is called ϑ . It is the map

$$\gamma_0(x \wedge y) = \sigma_2(x)\sigma_{-1}(y) - \sigma_{-1}(x)\sigma_2(y),$$

where the automorphism $\sigma_r \in \text{Gal}(K/\mathbb{Q}_3)$ is given by $\sigma_r(\theta) = \theta^r$. Lemma 5.1 of [11] shows that

$$\gamma_0(\mathfrak{p}^i \wedge \mathfrak{p}^j) = \mathfrak{p}^{i+j+\varepsilon} \quad \text{for} \quad \varepsilon = \begin{cases} 3 & i \equiv j \pmod{3} \\ 2 & \text{otherwise} \end{cases}.$$

Pick $j \geq 7$ and set $T = \mathfrak{p}^{j-3}$, $T_\ell = \mathfrak{p}^{j-3+\ell}$. Then $\gamma_0(T \wedge T) = T_j$, and $\gamma_0(T_j \wedge T) = T_k$ for

$$k = \begin{cases} 2j & 3 \mid j \\ 2j - 1 & 3 \nmid j \end{cases}.$$

Now pick a unit $c \in \mathcal{O}^\times$ and set $\gamma = c\gamma_0$. For any $m \in \{j, j+1, \dots, k\}$ one defines $T_{j-3, \gamma, m}$ to be the group with underlying set T/T_m and product

$$(x + T_m) * (y + T_m) = \left(x + y + \frac{1}{2}\gamma(x \wedge y) \right) + T_m.$$

Finally one sets $R_{j-3, \gamma, m} = T_{j-3, \gamma, m} \rtimes C$, where $C = \langle \tau \rangle$ has order 9 and acts on T via ${}^\tau v = \theta v$ for $v \in T$. Note that $|R_{j-3, \gamma, m}| = 3^{m+2}$.

Lemma 8.2. *Let $v, w \in T$.*

- (1) $(v + T_m)^r = rv + T_m$ in $T_{j-3, \gamma, m}$ for all $r \in \mathbb{Z}$.
- (2) *The order 3 elements of $R_{j-3, \gamma, m}$ are:*
 - *Elements of the form $(v + T_m)\tau^{3r}$, with $v \in T$ and $r \in \{1, 2\}$;*
 - *Elements of the form $v + T_m$ with $v \in T_{m-6}$.*
- (3) *If $v + T_m$ has order 3 then $\gamma(v \wedge w) \in T_m$ for all $w \in T$. Hence $\Omega_1(T_{j-3, \gamma, m}) \leq Z(T_{j-3, \gamma, m})$.*

Proof. (1): Follows by induction, since $\gamma(v \wedge rv) = r\gamma(v \wedge v) = 0$.

(2): Firstly, $[(v + T_m)\tau^3]^3 = (1 + \theta^3 + \theta^6)v + T_m = 0$. Secondly: $(v + T_m)^3 = 3v + T_m$. This is zero for $v \in 3^{-1}T_m = T_{m-6}$.

(3): Suppose that $v \in T_{m-6}$ and $w \in T$. Then $\gamma(v \wedge w)$ lies in $T_{j+m-9+\varepsilon}$, with $\varepsilon \in \{2, 3\}$. Since $\varepsilon \geq 2$ and $j \geq 7$, this means that $\gamma(v \wedge w)$ lies in T_m . \square

Lemma 8.3. (1) *The orbit of $(v + T_m)\tau^3$ under conjugation by $T_{j-3, \gamma, m}$ is*

$$\{(v' + T_m)\tau^3 \mid v \in v + T_3\}.$$

- (2) $(v + T_m)\tau^3$ and $(w + T_m)\tau^3$ are conjugate in $R_{j-3, \gamma, m}$ if and only if $v + T_3$ and $w + T_3$ lie in the same orbit under the action of C on T/T_3 .
- (3) *The action of $R_{j-3, \gamma, m}$ on T_{m-6}/T_m factors through C , and coincides with the action of C on T/T_6 via the isomorphism $v + T_6 \mapsto (\theta - 1)^{m-6}v + T_m$.*
- (4) $C_{T_{j-3, \gamma, m}}((v + T_m)\tau^3) = T_{m-3}/T_m$.

Proof. (1): Since T_m and the image of γ lie in $T_j \subseteq T_7$ we have

$$\begin{aligned} ({}^{w+T_m})[(v + T_m)\tau^3] &= [(w + T_m) * (v + T_m) * (-\theta^3 w + T_m)]\tau^3 \\ &\in (v + (1 - \theta^3)w + T_7)\tau^3. \end{aligned}$$

Since $\mathfrak{p} = (\theta - 1)\mathcal{O}$ and $3\mathcal{O} = \mathfrak{p}^6$ it follows that $(1 - \theta^3)T = (1 - \theta)^3T = \mathfrak{p}^3T = T_3$. So for each $v' \in v + T_3$ we find $w \in T$ with $({}^{w+T_m})[(v + T_m)\tau^3] = (v' + T_m)\tau^3$ and $v'' \in v' + T_7$. If we now adjust w by adding $u \in T_r$, then since $\gamma(T \wedge T_r) = T_{j+r-3+\varepsilon} \subseteq T_{r+6}$ we alter v' by an element of $(1 - \theta^3)u + T_{r+6}$. So if the error $v'' - v'$ lies in T_{s+3} , then with one correction we can reduce to an error in T_{s+6} . Iterating reduces the error to an element of T_m .

(2): Follows from (1). (3): The action factors by Lemma 8.2(3). The second statement follows, since C acts as multiplication by θ .

(4): Follows from (3), since T_3/T_6 is the subspace of T/T_6 consisting of elements fixed by τ^3 . \square

Let d be the number of orbits² for the action of C on T/T_3 . Pick $v_1, \dots, v_d \in T$ such that the $v_i + T_3$ form a set of orbit representatives for this action.

Lemma 8.4. *Every maximal elementary abelian subgroup of $R_{j-3,\gamma,m}$ is conjugate to precisely one of the following:*

- (1) d rank four groups of the form $V_i = \langle (v_i + T_m)\tau^3 \rangle \times T_{m-3}/T_m$;
- (2) $V_0 = T_{m-6}/T_m$ of rank six.

If $U \leq V_i$ is not contained in V_0 , then it is not conjugate to a subgroup of any other V_j .

Proof. Any elementary abelian outside T_{m-6}/T_m must contain some element of the form $(v + T_m)\tau^3$ and is therefore contained in $\langle (v + T_m)\tau^3 \rangle \times C_{T_{j-3,\gamma,m}}((v + T_m)\tau^3)$, that is $\langle (v + T_m)\tau^3 \rangle \times T_{m-3}/T_m$. Since $m \geq j \geq 7$ and therefore $m-3 \geq 3$, no two of the rank four elementary abelians in (1) are conjugate. This argument also demonstrates the last part. \square

Theorem 8.5. *Up to equivalence of categories, the Quillen category of the skeleton group $R_{j-3,\gamma,m}$ is independent of j, γ, m .*

Proof. V_0 is a normal subgroup, and Lemma 8.3(3) describes the conjugation action. So by the last part of Lemma 8.4 it suffices to show that if $U \leq V_i$ is not contained in V_0 , then the set of homomorphisms $U \rightarrow V_i$ lying in the Quillen category is independent of j, γ, m .

So $U = \langle (v + T_m)\tau^3 \rangle \times A$ for some $A \leq T_{m-3}/T_m$ and some $v \in v_i + T_{m-3}$. Consider conjugation by $(u + T_m)\tau^r$: by Lemma 8.3 this can only send $(v + T_m)\tau^3$ to an element of V if $\theta^r v_i$ lies in $v_i + T_3$; and if $\theta^r v_i$ does lie there, then by adjusting u we may send $(v + T_m)\tau^3$ to any element of the form $(v' + T_m)\tau^3$ with $v' \in v_i + T_{m-3}$. Moreover, the restriction to A of conjugation by $(u + T_m)\tau^r$ only depends on r . \square

8.3. The generalized quaternion groups. Let G be a finite group, and k a field of characteristic p . Write

$$\bar{H}^*(G, k) = \lim_{E \in \mathcal{A}_p(G)} H^*(E, k).$$

Quillen [16, Th. 6.2] proved that the induced homomorphism $\phi_G: H^*(G, k) \rightarrow \bar{H}^*(G, k)$ induces a homeomorphism between prime ideal spectra.

Our result shows that if G_r is a coclass family, then $\bar{H}^*(G_r, k)$ is independent of r . However this does *not* mean that the map ϕ_{G_r} is an isomorphism for large r . The (generalised) quaternion groups Q_{2^n} ($n \geq 3$) provide a good example.

²One easily verifies that $d = 11$.

The quaternion groups form a coclass sequence. The mod-2 cohomology ring $H^*(Q_{2^n}, \mathbb{F}_2)$ is well-known³:

$$\begin{aligned} H^*(Q_8, \mathbb{F}_2) &\cong \mathbb{F}_2[x, y, z]/(x^2 + xy + y^2, x^2y + xy^2) \\ H^*(Q_{2^n}, \mathbb{F}_2) &\cong \mathbb{F}_2[x, y, z]/(x^2 + xy, y^3) \quad (n \geq 4), \end{aligned}$$

with $x, y \in H^1$ and $z \in H^4$. Since $H^1(G, \mathbb{F}_2) = \text{Hom}(G, \mathbb{F}_2)$ and all order two elements lie in the Frattini subgroup, it follows that $x, y \in \ker(\phi_{Q_{2^n}})$. In fact $\bar{H}^*(Q_{2^n}, \mathbb{F}_2) \cong \mathbb{F}_2[z]$, since z restricts to the central C_2 as $t^4 \in H^*(C_2, \mathbb{F}_2) \cong \mathbb{F}_2[t]$: see Rusin's construction of z as a top Stiefel–Whitney class [17, p. 316]. So both $H^*(Q_{2^n}, \mathbb{F}_2)$ and $\bar{H}^*(Q_{2^n}, \mathbb{F}_2)$ are constant for $n \geq 4$, but $\phi_{Q_{2^n}}$ is never injective.

In fact one can demonstrate that $\phi_{Q_{2^n}}$ is never injective without even knowing the cohomology of Q_{2^n} . Recall that a class $x \in H^n(G, k)$ is called *essential* if its restriction to every proper subgroup $H < G$ vanishes: so if G is not elementary abelian, then every essential class lies in the kernel of ϕ_G . Now, Adem and Karagueuzian showed [1] that $H^*(G, \mathbb{F}_p)$ is Cohen–Macaulay and has non-zero essential elements if and only if G is a p -group and all order p elements are central. As Q_{2^n} satisfies this group-theoretic condition, it follows that $\ker(\phi_{Q_{2^n}}) \neq 0$.

REFERENCES

- [1] A. Adem and D. Karagueuzian. Essential cohomology of finite groups. *Comment. Math. Helv.*, 72(1):101–109, 1997.
- [2] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
- [3] D. J. Benson. *Representations and Cohomology. I*. Cambridge Studies in Advanced Math., vol. 30. Cambridge University Press, Cambridge, second edition, 1998.
- [4] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.
- [5] M. Couson. *On the character degrees and automorphism groups of finite p -groups by coclass*. PhD thesis, TU Braunschweig, 2013.
- [6] B. Eick. Automorphism groups of 2-groups. *J. Algebra*, 300(1):91–101, 2006.
- [7] B. Eick and D. Feichtenschlager. Computation of low-dimensional (co)homology groups for infinite sequences of p -groups with fixed coclass. *Internat. J. Algebra Comput.*, 21(4):635–649, 2011.
- [8] B. Eick and D. J. Green. The Quillen categories of p -groups and coclass theory. *Israel J. Math.*, 206(1):183–212, 2015.
- [9] B. Eick and S. King. The isomorphism problem for graded algebras and its application to mod- p cohomology rings of small p -groups. Submitted, Mar. 2015. arXiv:1503.04666 [math.RA].
- [10] B. Eick and C. Leedham-Green. On the classification of prime-power groups by coclass. *Bull. London Math. Soc.*, 40:274–288, 2008.

³To our knowledge the earliest references are [4, p. 253–4] for the additive structure and [15, p. 244] for the ring structure. By 1987, Rusin [17, p. 316] could quote the result without needing to give a reference.

- [11] B. Eick, C. R. Leedham-Green, M. F. Newman, and E. A. O'Brien. On the classification of groups of prime-power order by coclass: the 3-groups of coclass 2. *Internat. J. Algebra Comput.*, 23(5):1243–1288, 2013.
- [12] L. Evens. *The cohomology of groups*. Oxford Univ. Press, Oxford, 1991.
- [13] C. R. Leedham-Green and S. McKay. *The structure of groups of prime power order*, volume 27 of *London Mathematical Society Monographs. New Series*. Oxford University Press, Oxford, 2002. Oxford Science Publications.
- [14] C. R. Leedham-Green and M. F. Newman. Space groups and groups of prime-power order. I. *Arch. Math. (Basel)*, 35(3):193–202, 1980.
- [15] H. J. Munkholm. Mod 2 cohomology of $D2^n$ and its extensions by Z_2 . In *Conf. on Algebraic Topology (Univ. of Illinois at Chicago Circle, Chicago, Ill., 1968)*, pages 234–252. Univ. of Illinois at Chicago Circle, Chicago, Ill., 1969.
- [16] D. Quillen. The spectrum of an equivariant cohomology ring. I, II. *Ann. of Math. (2)*, 94: 573–602, 1971.
- [17] D. J. Rusin. The mod-2 cohomology of metacyclic 2-groups. *J. Pure Appl. Algebra*, 44(1-3):315–327, 1987.

INSTITUT COMPUTATIONAL MATHEMATICS, TU BRAUNSCHWEIG, 38106 BRAUNSCHWEIG, GERMANY

E-mail address: `beick@tu-bs.de`

INSTITUTE FOR MATHEMATICS, FRIEDRICH-SCHILLER-UNIVERSITÄT JENA, 07737 JENA, GERMANY

E-mail address: `david.green@uni-jena.de`